

# Two- and three-body structure of the $Y(4660)$

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## Abstract

We study general features of three-meson bound states using the  $Y(4660)$  as an example. Here the  $Y(4660)$  is assumed to be either a two-body bound state of the  $f_0(980)$ , itself a bound state of  $K$  and  $\bar{K}$ , and the  $\psi' = \psi(2s)$ , or a three-body bound state of  $\psi'$ ,  $K$ , and  $\bar{K}$ . In particular, we investigate in detail the interplay of the various scales inherent in the problem, namely the  $f_0$  binding energy, the  $Y$  binding energy, and the  $K\psi'$  scattering length. This allows us to understand under which circumstances the substructure of the  $f_0(980)$  can be neglected in the description of the  $Y(4660)$ .

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**1. Introduction:** In recent years a large number of new charmonium states has been discovered experimentally, most of them showing properties in vast conflict with the predictions of the quark model [1]. The structure of these states presents a serious challenge for our understanding of QCD. Since many of these new states are located near thresholds, it appeared natural to propose a molecular nature for those. The prime example is the  $X(3872)$  located right at the  $\bar{D}^0 D^{*0}$  threshold, proposed to be a molecule composed of these mesons [2] or at least a virtual state [3]. However, this picture is not uniformly accepted. It was, for example, challenged on the basis of production data from  $\bar{p}p$  [4]. A rebuttal to this challenge was given in Ref. [5]. In the light quark sector, the light scalar mesons located below 1 GeV are also proposed to be molecules, namely the  $f_0(600)$  — the  $\sigma$  meson — the  $f_0(980)$ ,  $a_0(980)$ , and  $\kappa(900)$  [6,7]. Because of the relatively large widths involved, especially for the  $\sigma$ , and the larger distance to the scattering thresholds, however, the situation is less clear in this case. For a different viewpoint on the structure of these states, see [8,9].

These examples illustrate the importance of a model-independent method to

determine the structure of such states close to scattering thresholds. For two-body states located near a threshold there is a powerful method originally proposed by Weinberg for the deuteron [10]. This method was extended in Ref. [11] and allows one to quantify the two-body molecular component. A complementary formulation of this problem is given by effective range theory and the effective field theory (EFT) for large scattering length. The latter framework can also be extended to three- and four-body molecules [12,13]. Both methods effectively analyze the low-energy pole structure of the scattering amplitude in a model-independent way.

In this paper, we want to investigate to which extent it is possible to distinguish between two- and three-body molecules and which scales govern this distinction. In particular, we analyze the example of the  $Y(4660)$  in detail. This state was proposed to be a  $\psi' f_0(980)$  molecule based on an analysis with the Weinberg method [14] while the  $f_0(980)$  itself was proposed to be a molecule of  $\bar{K}$  and  $K$  mesons [6,7]. It is therefore important to elucidate whether the internal structure of the  $f_0(980)$  can be neglected for the description of the  $Y(4660)$  or not. The answer to this question, of course, depends on the physical scales in this problem. For a deeply-bound  $f_0(980)$ , one expects that the internal structure can be neglected. Performing numerically exact three-body calculations, we determine for which parameter ranges the  $Y(4660)$  can be interpreted as a  $\psi' f_0(980)$  molecule and for which ranges as a  $\psi' K \bar{K}$  molecule. We stress that we only investigate the interplay of a possible two-body vs. three-body nature of the  $Y(4660)$  in order to understand the interplay of the various scales. No explicit compact component is included for the  $Y(4660)$  nor is the width of the  $f_0(980)$  included. Thus, this work is mainly of theoretical interest and we are not yet in the position to compare with experimental line shapes. We start with a brief description of our formalism and some analytical considerations before we present and discuss the results of our three-body calculation. The paper ends with an outlook.

**2. Formalism:** In this section, we briefly outline the approach used to derive the amplitude for  $\psi' f_0(980)$ -scattering. For this purpose, we set up a non-relativistic effective field theory (EFT) for three distinguishable particles with different masses.<sup>1</sup> The underlying Lagrangian reads [15]

$$\begin{aligned} \mathcal{L} = & \sum_{k=0}^2 \psi_k^\dagger \left( i\partial_t + \frac{\vec{\nabla}^2}{2m_k} \right) \psi_k - \sum_{k=0}^2 g_k \left( d_k^\dagger d_k - d_k^\dagger \psi_i \psi_j - d_k \psi_i^\dagger \psi_j^\dagger \right) \\ & - h \sum_{k=0}^2 c_k d_k^\dagger d_k \psi_k^\dagger \psi_k \quad , \end{aligned} \tag{1}$$

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<sup>1</sup> A detailed description of the EFT for three identical bosons is given in [13]. The general case of distinguishable particles with different masses can be found in Ref. [15].

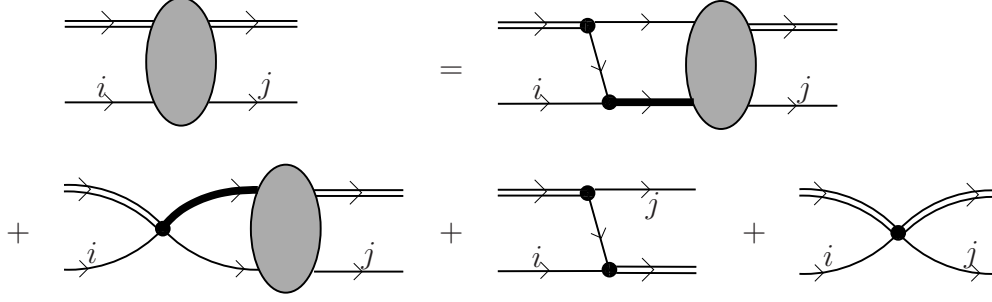


Fig. 1. Diagrammatic representation of the integral equations used. The bare and full dimer fields are indicated by the double and thick lines, respectively. The equation for the latter is illustrated in Fig. 2.

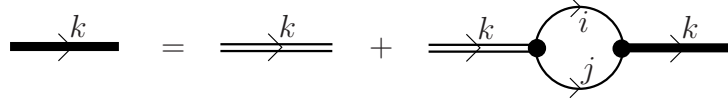


Fig. 2. Diagrammatic representation for the full dimer propagator in channel  $k$ .

where the scalar functions  $\psi_k$  and  $d_k$  are particle- and dimer fields, respectively, and  $m_k$  is the mass of particle  $k$ . The indices  $i, j, k$  in Eq. (1) are always different from each other such that the mass of particle  $k$  can also be labeled by  $m_{ij} = m_{ji}$ . This convention will be used below. Moreover, a two-body state of particles  $i$  and  $j$  can be labeled either by the indices  $ij$  or by the index  $k$  ( $i \neq j \neq k$ ). The dimer fields are introduced for convenience and are not dynamical. An equivalent theory without dimer fields can be obtained by inserting the classical equation of motion for the non-dynamical dimer fields. Furthermore,  $g_k$  and  $h$  are coupling-constants, and the factors  $c_k$  sum up to one. From the Lagrangian (1), we can directly derive the Feynman-rules of our theory in momentum space and construct the general amplitude  $A_{ij}$  for particle-dimer scattering in the channel  $i \rightarrow j$ . The resulting equations are illustrated in Figs. 1 and 2.

After renormalizing the amplitude and performing a partial wave decomposition, the inhomogeneous integral equation for the  $l$ -th partial wave reads

$$[A_{ij}]_l(p, k) = \sum_{k=0}^2 \int_0^\Lambda dq [T_{ik}]_l(p, q) S_k(q) [A_{kj}]_l(q, k) - [T_{ij}]_l(p, k) \quad , \quad (2)$$

where  $p$  and  $k$  is the incoming and outgoing momentum, respectively. The kernel factorizes into the driving term

$$[T_{ik}]_l(p, q) := (1 - \delta_{ik}) \frac{2\pi m_{ik}}{\mu_i \mu_k \sqrt{|a_i|} |a_k|} \frac{1}{pq} Q_l(c_{ik}(p, q)) + \delta_{ik} \delta_{l0} \frac{H(\Lambda)}{\Lambda^2} \quad (3)$$

and the full dimer propagator

$$S_k(q) := \frac{\mu_k |a_k| q^2}{2\pi^2} \left[ \frac{1}{a_k} - \sqrt{2\mu_k \left( \frac{q^2}{2\tilde{\mu}_k} - E \right) - i\epsilon} \right]^{-1}, \quad (4)$$

where

$$\mu_k := \frac{m_i m_j}{m_i + m_j} \quad \text{and} \quad \tilde{\mu}_k := \frac{(m_i + m_j) m_k}{(m_i + m_j) + m_k} \quad (5)$$

are the reduced masses of the two-particle system labeled by  $ij$  or by  $k$  ( $i \neq j \neq k$ ) and the particle-dimer system labeled by  $k$ , respectively. The Legendre functions of the second kind  $Q_l$  and their arguments  $c_{ik}(p, q)$  are defined by

$$Q_l(c) := \frac{1}{2} \int_{-1}^1 dx \frac{P_l(x)}{c - x} \quad \text{and} \quad c_{ik}(p, q) := \frac{m_{ik}}{pq} \left( E - \frac{p^2}{2\mu_k} - \frac{q^2}{2\mu_i} + i\epsilon \right), \quad (6)$$

where  $P_l$  denotes the  $l$ -th Legendre polynomial. We use standard solution techniques in order to solve the integral Eq. (2) numerically. The apriori undetermined, dimensionless function  $H(\Lambda)$ , that by construction contributes only in the  $s$ -wave, is directly related to the parameter  $h$  in Eq. (1). The function  $H(\Lambda)$  plays the role of a running coupling constant and is determined by the renormalization procedure up to an unknown three-body parameter [16]. We fix its value by imposing that a three-particle bound-state exists at the binding energy  $E = -B$ . This requires that the amplitude  $A_{ij}$  in Eq. (2) has a pole at this energy. However, any low-energy three-body observable can be chosen for this purpose. After fixing  $H$ , we are able to calculate the full particle-dimer scattering amplitude and therefrom, observables such as scattering lengths, cross sections and so forth can be predicted.

This theory represents the leading order of an EFT around the limit of large pair scattering lengths  $a_k$  [12,13]. Corrections are suppressed by powers of  $r_0/a_k$  and  $pr_0$ , where  $r_0$  is the range of the interaction and  $p$  is a typical momentum. If the scattering lengths are not sufficiently large, the theory can still be taken as a particular three-body model for hadronic molecules. As such, it can be used to test the limits of applicability of the Weinberg approach to systems like the  $Y(4660)$ . By calculating the scattering length and the effective range, we are able to investigate the question, whether the three-particle bound-state can effectively be treated as a two-particle system, consisting of one particle and an elementary dimer. If the substructure of the dimer becomes relevant, the full three-particle picture is necessary. Transferred to our specific example  $Y(4660)$ , this means that we want to distinguish between the two alternatives  $\psi' f_0(980)$  and  $\psi' K \bar{K}$ . In the Weinberg formalism [10], the quantity  $Z \in [0, 1]$  is defined as

$$Z := 1 - \int dp \left| \langle \psi' f_0(p) | Y \rangle \right|^2, \quad (7)$$

where  $Y$  denotes the wave function of the physical state. Thus,  $Z$  measures the "non-two-particle"-fraction of the wave function. From the definition (7), we directly conclude that  $Z \rightarrow 0$  implies that the  $Y(4660)$  is effectively a two-body system with an elementary  $f_0(980)$ -dimer, whereas  $Z \rightarrow 1$  means that it has to be seen as something else, in our case a three-body molecule. This method of distinction between two- and three-particle molecules can of course also be applied to other candidates for hadronic molecules in order to reveal their few-particle nature. The quantity  $Z$  is directly related to the residue  $Z_{pole}$  of the bound state pole in the two-particle scattering amplitude by  $Z = 1 - Z_{pole}$ .

Performing a straightforward calculation within the Weinberg formalism, using the Lippmann-Schwinger equation and the effective range expansion, the quantity  $Z$  can be related to the effective range parameters in two different ways [10]. The two corresponding  $Z$ -factors, which we denote by

$$Z_a := \frac{a_{\psi'f_0}^{2P} - a_{\psi'f_0}}{a_{\psi'f_0}^{2P} - \frac{a_{\psi'f_0}}{2}} \quad \text{and} \quad Z_r := \frac{r_{\psi'f_0}}{r_{\psi'f_0} - a_{\psi'f_0}^{2P}}, \quad (8)$$

should be (approximately) equal if the Weinberg formalism is applicable. The equations (8, 9) receive corrections from the finite range of the interaction. They are exact in the limit of vanishing binding energy of the  $Y(4660)$  relative to the  $\psi'f_0$ -threshold with  $Z = Z_a = Z_r$  kept fixed [10]. The quantities  $a_{\psi'f_0}$  and  $r_{\psi'f_0}$  are the scattering length and the effective range for  $\psi'f_0$ -scattering, respectively. Moreover,

$$a_{\psi'f_0}^{2P} = \frac{1}{\sqrt{2 \mu_{\psi'f_0} B_{\psi'f_0}}} = \frac{1}{\sqrt{2 \mu_{\psi'f_0} (B_Y - B_{f_0})}} \quad (9)$$

is the scattering length within a pure two-particle picture.

The positive binding energies are defined via

$$B_Y := m_{\psi'} + m_K + m_{\bar{K}} - m_Y \quad (10)$$

$$B_{\psi'f_0} := m_{\psi'} + m_{f_0} - m_Y \quad (11)$$

$$B_{f_0} := m_K + m_{\bar{K}} - m_{f_0} \quad . \quad (12)$$

Thereby all quantities on the right sides of the equations in (8) can be calculated in the full three-body model. Combining the two foregoing conditions  $Z \in [0, 1]$  and  $Z = Z_a = Z_r$ , we conclude that

$$0 \leq Z_a \approx Z_r \leq 1 \quad (13)$$

should hold. Below we will use the validity of Eq. (13) as a diagnostic tool to identify the range of applicability of the Weinberg method.

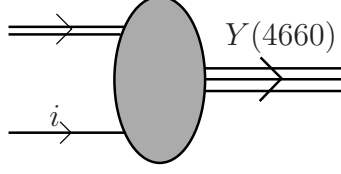


Fig. 3. Feynman diagram for the form-factor  $\langle d_i \psi_i(p) | \hat{V} | Y \rangle$  with a particle and an dimer of type  $i$  in the incoming channel and  $Y(4660)$  as the outgoing bound state.

The full derivation of the equations (8) can be found in [10]. We omit it at this point, but outline its essential idea. Utilizing the binding energy between the particle  $\psi'$  and  $f_0(980)$  as defined in Eq. (11), the Schrödinger equation can be applied in the form

$$\hat{H} | Y \rangle = -B_{\psi' f_0} | Y \rangle \quad \text{with} \quad \hat{H} = \hat{H}_0 + \hat{V} \quad . \quad (14)$$

Here  $\hat{V}$  describes the interaction and  $\hat{H}_0$  denotes the free-particle part of the Hamiltonian, such that

$$\hat{H}_0 | \psi' f_0(p) \rangle = E(p) | \psi' f_0(p) \rangle \quad (15)$$

holds, where  $E(p) = p^2/(2\mu_{\psi' f_0})$  and  $p$  is the relative momentum of the  $\psi'$  and  $f_0$  in the center-of-mass. Employing the relations (14) and (15), the momentum-integral appearing in the definition of  $Z$  (7) assumes the form

$$\begin{aligned} \int dp \left| \langle \psi' f_0(p) | Y \rangle \right|^2 &= \int dp \left| \langle \psi' f_0(p) | \frac{\hat{H}_0 - \hat{H}}{E(p) + B_{\psi' f_0}} | Y \rangle \right|^2 \\ &= \int dp \frac{|\langle \psi' f_0(p) | \hat{V} | Y \rangle|^2}{(E(p) + B_{\psi' f_0})^2} \quad . \end{aligned} \quad (16)$$

In order to deduce the formulas (8), the crucial requirement of the Weinberg method is that the numerator on the right-hand-side of Eq. (16) can be approximated by its value at zero momentum,

$$\int dp \frac{|\langle \psi' f_0(p) | \hat{V} | Y \rangle|^2}{(E(p) + B_{\psi' f_0})^2} \approx \int dp \frac{|\langle \psi' f_0(0) | \hat{V} | Y \rangle|^2}{(E(p) + B_{\psi' f_0})^2} \quad . \quad (17)$$

For this to hold, the numerator has to vary slowly over the range of momenta contributing to the integral, i.e.

$$E(p) \lesssim B_{\psi' f_0} \quad \Rightarrow \quad p \lesssim \sqrt{2\mu_{\psi' f_0} B_{\psi' f_0}} \quad . \quad (18)$$

In other words, the range of the form factor  $\langle \psi' f_0(p) | \hat{V} | Y \rangle$  has to be much larger than the range of the denominator,  $\sqrt{2\mu_{\psi' f_0} B_{\psi' f_0}}$ .

**3. Analytical Considerations:** Thus we have to understand the relevant momentum-scales of  $\langle \psi' f_0(p) | \hat{V} | Y \rangle$ . In the general case, with an incoming

particle and dimer of type  $i$  this form-factor  $\langle \psi_i d_i(p) | \hat{V} | Y \rangle$  is proportional to the bound state amplitude  $F_i$  as it is depicted in Fig. 3. It can be calculated by solving the homogeneous analog of Eq. (2),

$$[F_i]_0(p) = \sum_{k=0}^2 \int_0^\Lambda dq [T_{ik}]_0(p, q) S_k(q) [F_k]_0(q) \quad , \quad (19)$$

where we have set  $l = 0$  for an  $s$ -wave state. In the rest frame of the  $Y(4660)$ , the total energy is simply  $E = -B_Y$ . The quantity  $p$  is the relative momentum between the incoming particles. Solving Eq. (19) exactly requires the same effort as solving the scattering equation and can only be done numerically. Such numerical solutions will be presented below.

Our aim here is to investigate to what extent the underlying scales of the form factor  $\langle \psi_i d_i(p) | \hat{V} | Y \rangle$  can be understood in a simple way. For this purpose, we approximate the full amplitude  $[F_k]_l(q)$  on the right-hand-side of Eq. (19) by a constant. In this point-like approximation, the  $p$ -dependence of  $\langle \psi_i d_i(p) | \hat{V} | Y \rangle$  emerges from the integral

$$\begin{aligned} & \int_0^\Lambda dq [T_{ik}]_0(p, q) S_k(q) \\ &= \int_0^\Lambda dq \left\{ (1 - \delta_{ik}) \frac{m_{ik}}{\pi \mu_i} \sqrt{\frac{|a_k|}{|a_i|}} \frac{q}{p} Q_0(c_{ik}(p, q)) + \delta_{ik} \frac{H(\Lambda)}{\Lambda^2} \frac{\mu_k |a_k| q^2}{2\pi^2} \right\} \\ & \quad \times \left[ \frac{1}{a_k} - \sqrt{2\mu_k \left( B_Y + \frac{q^2}{2\tilde{\mu}_k} \right)} \right]^{-1} \quad , \end{aligned} \quad (20)$$

where we have inserted the definitions (3) and (4). Using (6), we deduce that for  $E = -B_Y$  the quantity  $c := c_{ik}(p, q)$  is always smaller than  $-1$  and approaches  $-\infty$  for  $p \rightarrow 0$ . Hence a Taylor-expansion of  $Q_0$  around  $1/c = 0$  can be performed and we approximate:

$$Q_0(c) = \frac{1}{2} \ln \frac{1 + 1/c}{1 - 1/c} \approx \frac{1}{c} \quad . \quad (21)$$

Inserting this result into (20), the momentum dependence of  $\langle \psi_i d_i(p) | \hat{V} | Y \rangle$  for  $p \rightarrow 0$  has the form:

$$\propto (1 - \delta_{ik}) \int_0^\Lambda dq \frac{q^2}{B_Y + \frac{p^2}{2\mu_k} + \frac{q^2}{2\mu_i}} \frac{1}{\frac{1}{a_k} - \sqrt{2\mu_k \left( B_Y + \frac{q^2}{2\tilde{\mu}_k} \right)}} \quad . \quad (22)$$

This expression is nearly constant for momenta

$$0 \leq p \lesssim \sqrt{2\mu_k B_Y} \quad , \quad (23)$$

so that the range of the form-factor can be estimated by  $\sqrt{2\mu_k B_Y}$ . Coming back to our specific situation with  $\psi'$  and the  $f_0$ -dimer, the range of  $\langle \psi' f_0(p) | \hat{V} | Y \rangle$  is estimated as  $\sqrt{2\mu_{\psi' K} B_Y}$ . From this we are able to formulate the condition

$$\sqrt{2\mu_{\psi' K} B_Y} \gg \sqrt{2\mu_{\psi' f_0} B_{\psi' f_0}} \quad (24)$$

for the validity of Eq. (17). Since the reduced mass terms  $\mu_{\psi' K}$  and  $\mu_{\psi' f_0}$  are both of the order  $m_K$  and  $B_{\psi' f_0} = B_Y - B_{f_0}$ , this is equivalent to:

$$B_Y \gg B_Y - B_{f_0} \quad \Leftrightarrow \quad 1 \gg 1 - b_0 \quad , \quad (25)$$

where we have introduced the dimensionless parameter  $b_0 := B_{f_0}/B_Y \in (0, 1)$ , which is the binding energy of the  $f_0$ -dimer relative to the binding energy of the  $Y(4660)$ . Thus in this approximation the only relevant scale that controls the applicability of the Weinberg-method is  $b_0$  which has to be close to 1, meaning that  $f_0$  has to be a deeply bound state or an elementary particle.

We stress that  $b_0$  can not be the only relevant scale in this problem. In particular, the scale characterizing the  $K\psi'$  and  $\bar{K}\psi'$  interactions does not enter at all in this simple picture. In the full solution of the three-body problem where the  $Y(4660)$  emerges as a dynamically generated three-particle-state, the situation will be more complex. We will investigate this question below by solving Eq. (2) numerically.

**4. Results and discussion:** Using our formalism, we now present some results for  $\psi' f_0(980)$ -scattering within our three-body approach for the  $Y(4660)$ . In order to calculate observables, we first have to fix the 3 masses and scattering lengths, appearing in (3), (4) and (6). Ignoring the – for our purpose – insignificant widths and errors, we take PDG values for the 3 masses by setting  $m_{\psi'} = 3686.1$  MeV and  $m_K = m_{\bar{K}} = (m_{K^+} + m_{K^0})/2 = 495.6$  MeV. The corresponding two-particle scattering lengths are currently unknown, but for the kaon-kaon-interaction we can determine  $a_{K\bar{K}} = 1/\sqrt{2\mu_{K\bar{K}} B_{f_0}}$  by employing the analogue of Eq. (9). This corresponds to treating the  $f_0$  as shallow bound state of  $K$  and  $\bar{K}$ . For  $a_{\psi' K}$  and  $a_{\psi' \bar{K}}$ , we have neither experimental data nor predictions from lattice calculations. Due to the absence of a two-particle bound-state and owing to symmetry reasons we can, however, conclude that  $a_{\psi' K} = a_{\psi' \bar{K}} < 0$  should hold. Another restriction on this quantity's magnitude comes from the fact that in  $\psi' K$ -scattering no quark exchange is possible in contrast to  $DK$ -scattering, for instance. Therefore, its interaction should be suppressed compared to the one in the latter case, implying that  $|a_{\psi' K}|$  should be smaller than typical, non-resonant scattering lengths for  $DK$ -processes. In [17],  $DK$  scattering lengths were calculated, using unitarised chiral perturbation theory and in Ref. [18] they were extracted from lattice QCD simulations. Both analyses agree and provide values of the order of 0.1 fm. Thus we demand  $-0.1 \text{ fm} \lesssim a_{\psi' K} = a_{\psi' \bar{K}} < 0 \text{ fm}$ .



We note that, given this assumption,  $a_{\psi'K}$  is not large compared to the range of forces such that the EFT for large scattering lengths is strictly not applicable. In this study, however, we use the EFT as a specific three-body model that can be solved numerically in order to test the range of applicability of the Weinberg method. In relation to the corresponding thresholds in Eq. (10) and (12) the masses  $m_Y = 4664 \pm 11 \pm 5$  MeV and  $m_{f_0} = 980 \pm 10$  MeV have large errors respectively so that we are still allowed to vary the binding energies in a region  $0 < B_{f_0} < B_Y \lesssim 20$  MeV and be consistent with experiment.

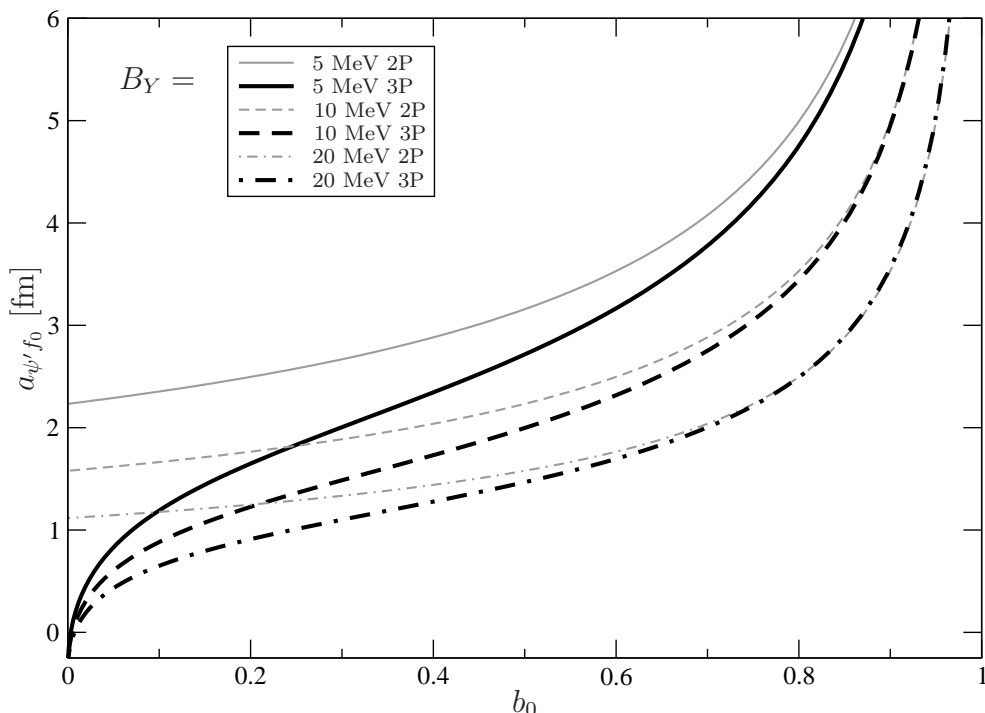


Fig. 4. Scattering length for the  $\psi'f_0(980)$ -process as a function of the relative binding energy  $b_0 = B_{f_0}/B_Y$  for different values of  $B_Y$ , the binding energy of the  $Y$  with respect to the  $\bar{K}K\psi'$  threshold (see Eq. (11)). Our numerical values within the three-particle model (3P) are compared to the ones in a simple two-particle model (2P). The unknown scattering length  $a_{\psi'K} = a_{\psi'\bar{K}}$  is set to  $-0.1$  fm.

In Fig. 4, the numerically calculated  $\psi'f_0(980)$ -scattering length within the three-particle model (3P) is depicted for several values of the total binding energy  $B_Y$ . It is also compared to the one of Eq. (9) in a hypothetical two-particle system (2P) with no substructure in the  $f_0(980)$ -dimer. The variation in  $B_{f_0}$  is expressed in a dimensionless relative binding energy  $b_0$ . The unknown quantity  $a_{\psi'K} = a_{\psi'\bar{K}}$  is set to  $-0.1$  fm.<sup>2</sup> Our numerical results for the  $\psi'f_0(980)$ -scattering length confirm the behavior deduced from the analytical considerations above (cf. Eq. (25)). For a weakly bound  $f_0(980)$ -dimer, that

<sup>2</sup> For the numerical calculation, we typically use  $N = 100 - 200$  mesh points and a cutoff  $\Lambda = 10^{11}$  MeV. The obtained results are independent of this choice, however, as long as  $\Lambda$  is large compared to all momentum scales in the problem.

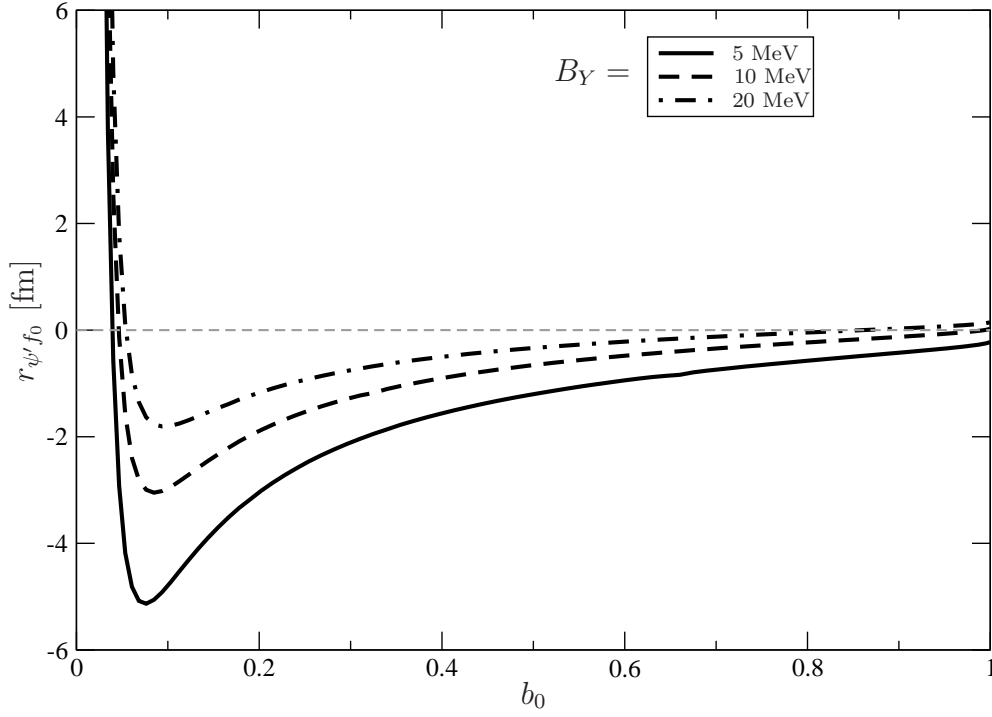


Fig. 5. Effective range for  $\psi' f_0(980)$ -scattering as a function of the relative binding energy  $b_0 = B_{f_0}/B_Y$  for different values of  $B_Y$ . The unknown scattering length  $a_{\psi' K} = a_{\psi' \bar{K}}$  is set to  $-0.1$  fm.

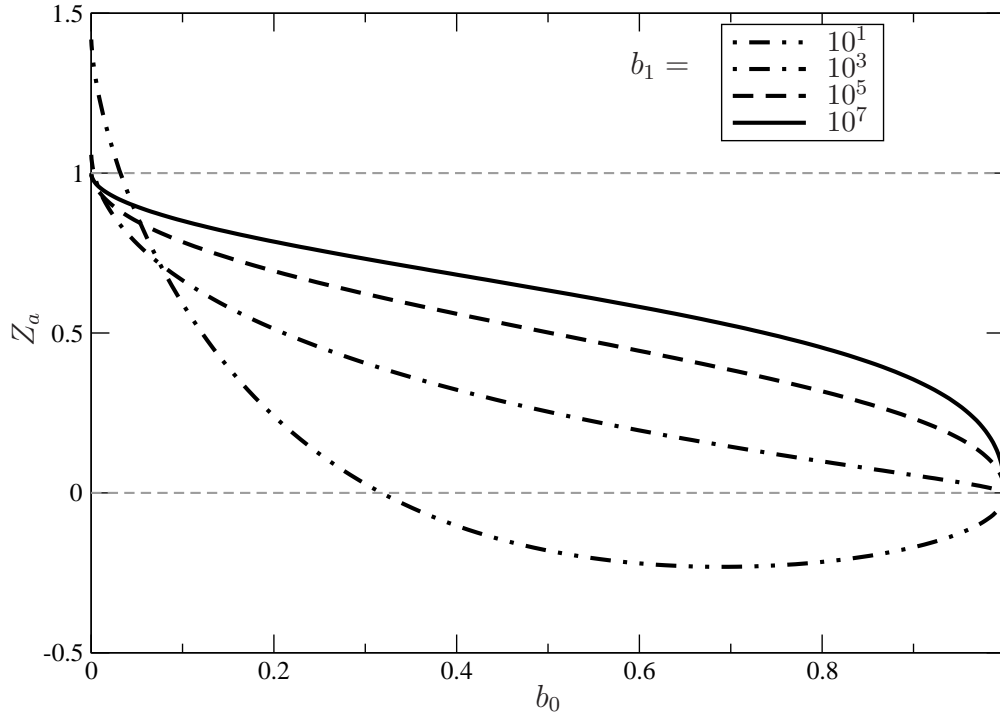


Fig. 6.  $Z_a$  from Eq. (8) as a function of the relative binding energy  $b_0 = B_{f_0}/B_Y$  for different values of  $b_1 = B_1/B_Y$ .

is  $b_0 \rightarrow 0$ , we see deviations between the two models, whereas for a strongly bound dimer with  $b_0 \rightarrow 1$  the results nearly coincide. In the latter case  $a_{\psi' f_0}$  diverges to  $+\infty$ . We also calculated the effective range in Fig. 5 for the same parameters. It diverges at values where the scattering length vanishes. Unfortunately, neither experimental data nor lattice calculations are available at present and no comparison to data for  $\psi' f_0(980)$ -scattering can be made.

By applying the Weinberg-formalism, described in the previous section, we are also able to investigate the question at which scales  $Y(4660)$  can effectively be seen as a two-body molecule consisting of elementary particles  $f_0(980)$  and  $\psi'$  and where the  $f_0(980)$ -substructure has to be taken into account. Since the only undetermined scales in our system are  $B_Y$ ,  $B_{f_0}$  and  $a_{\psi' K}$ , we expect the dimensionless quantities  $Z_a$  and  $Z_r$ , as defined in the equations (8), only to depend on 2 dimensionless ratios of these 3 parameters. We have already shown above that the relative binding energy of the  $f_0$  and  $Y(4660)$ ,  $b_0$ , plays an important role in determining the structure of the  $Y(4660)$ . For the second ratio it is natural to choose  $b_1 := B_1/B_Y := 1/(2\mu_{\psi' K} a_{\psi' K}^2 B_Y)$ . Since the scattering length  $a_{\psi' K}$  is negative, the energy scale  $B_1$  characterizes a virtual state in the  $\psi' K$ -channel but not a bound state. The functions  $Z_a(b_0)$  for discrete values of  $b_1$ , varied over a wide range, are displayed in Fig. 6. We find that  $Z_a$  strongly depends on  $b_0$  and  $b_1$ . For  $b_0 \rightarrow 0$ ,  $Z_a$  reaches the value of 1 so that  $Y(4660)$  has to be considered a three-body system. In the limit  $b_0 \rightarrow 1$ ,  $Z_a$  approaches 0 as the  $f_0(980)$ -dimer gets maximally bound. In this case, the  $Y(4660)$  can be seen as a two-particle system consisting of the elementary particles  $\psi'$  and  $f_0(980)$ .

Another feature shown in Fig. (6) is the fact that for certain parameters  $Z_a$  exceeds its allowed interval  $[0, 1]$ . This leads us back to question at which scales the initial approximation (17) is valid. The approximate condition (25) for  $b_0$  and the exact condition for  $b_0$  and  $b_1$  determined using Eq. (13) as a consistency-check for the theory, are summarized in Fig. 7. We show the consistency conditions for the Weinberg method in the  $b_0$ - $\sqrt{b_1}$ -plane.<sup>3</sup> The shaded grey area indicates the allowed region with  $0 < Z_a < 1$ , while the area to the right of the dotted horizontal line satisfies the consistency condition  $Z_a \approx Z_r$ . The area to the right of the dashed horizontal line satisfies the approximate condition (25):  $1 \gg 1 - b_0$ . Within the upper right corner (dashed area) both our approximate and numerically exact calculations indicate the applicability of the Weinberg method. Outside of this area it leads to inconsistent results and can not be applied. For the case of the  $Y(4660)$ , the Weinberg method is applicable in a significant part of the physical region of interest. This is due to the fact that the experimental/theoretical constraints  $B_Y \lesssim 20$  MeV and  $|a_{\psi' K}| \lesssim 0.1$  fm restrict the parameter  $b_1 = 1/(2\mu_{\psi' K} a_{\psi' K}^2 B_Y)$  to be

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<sup>3</sup> The dimensionless quantity  $b_1$  has been replaced by its square root to magnify the most relevant parameter region.

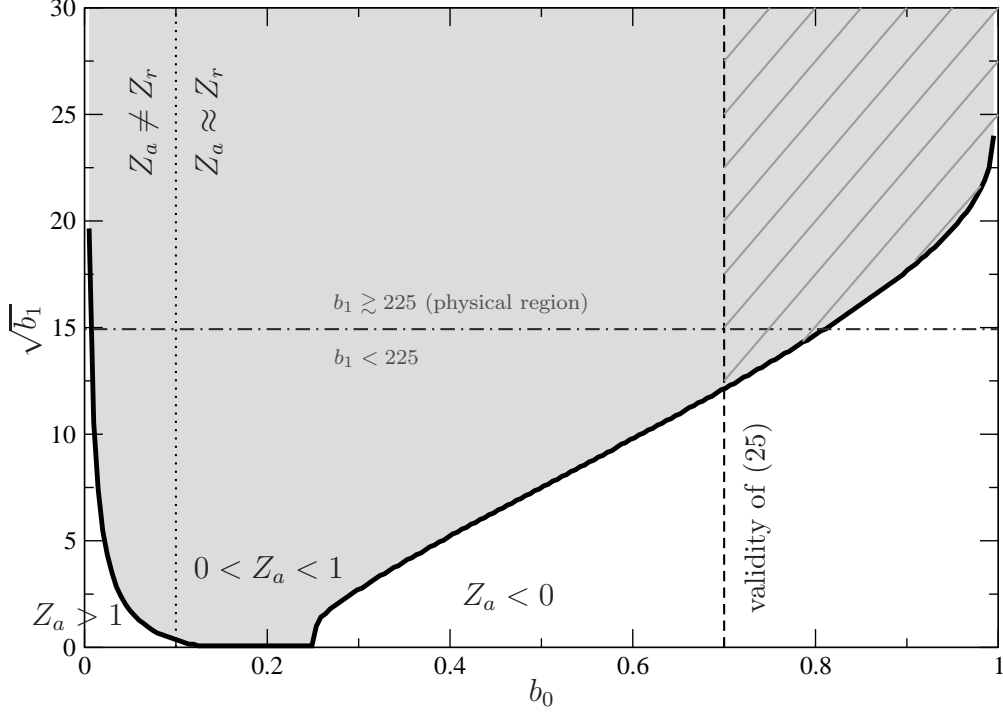


Fig. 7. Consistency conditions for the Weinberg method in the  $b_0$ - $\sqrt{b_1}$ -plane. The shaded grey area indicates  $0 < Z_a < 1$ , while the area to the right of the dotted horizontal line satisfies  $Z_a \approx Z_r$ . The area to the right of the dashed horizontal line satisfies the approximate condition (25) for  $b_0$ . In the overlap of these regions indicated by the dashed area, the Weinberg formalism is applicable. This area also lies within the physical region of interest for the  $Y(4660)$  ( $b_1 \gtrsim 225$ ).

greater than approximately 225.

**5. Summary and Outlook:** In this work, we have investigated whether it is possible to distinguish between two- and three-body molecules and which scales govern this distinction. We identified the ratios of the characteristic energies of the two-body subsystems to the characteristic three-body energy  $b_0$  and  $b_1$  as the relevant scales. Moreover, we found a parameter region where the two-body picture is appropriate and other regions where it leads to inconsistencies. In particular, we have applied our formalism to the example of the  $Y(4660)$ . There, we found that a two-body picture is applicable in a significant part of the relevant region in the parameter space.

We also note that our work can not yet be used to compare to experimental line shapes, especially since we assumed the  $f_0(980)$  as a stable particle. In reality, due to the decay  $f_0(980) \rightarrow \pi\pi$  the  $Y(4660)$  becomes observable in  $\psi'\pi\pi$  invariant mass distributions and the resulting line shapes are believed to contain important information on the nature of the  $Y$  [14]. Within the formalism presented, this channel could be included via a complex scattering length of the  $\bar{K}K$  system — the impact of unstable constituents on the line shapes of particles is, e.g., discussed in Ref. [19]. In addition, the  $Y(4660)$  may

also decay into other channels. Currently there is a discussion, if the signal seen in  $\Lambda_c \bar{\Lambda}_c$ , baptized  $X(4630)$ , has its origin in the  $Y(4660)$  [20,21,22] — this channel could be parametrized by an imaginary part of the three-body interaction.

However, we do not expect that the possible extensions mentioned will distort significantly the conclusions on the applicability of the Weinberg method. Our analytical analysis summarized in Eq. (25) shows that the binding energy of the  $Y(4660)$  relative to the three-body threshold has to be large compared to the molecular binding energy relative to the  $\psi' f_0$  threshold,  $1 \gg 1 - b_0$ . Moreover, we found that if the scattering length in the other subsystems is large, this kind of analysis can not be used and should either be replaced by a more complex, coupled channel analysis or abandoned all together. However, in a significant part of the parameter space allowed for the  $f_0(980)$  and the  $Y(4660)$  shown in Fig. 7, the Weinberg method can be used to quantify the two-body molecular component. Because of the sizeable errors in the masses of the  $Y(4660)$  and the  $f_0(980)$  and the unknown  $\psi' K$  scattering length, no definite conclusion about the nature of the  $Y(4660)$  can be reached at present.

Various extensions of our approach are possible. In principle, we could also integrate spin-effects, higher derivatives in the fields and, via photon-coupling, even charge-dependent interactions into the Lagrangian (1). Clearly all these extensions imply additional parameters in our theory which would have to be determined. Furthermore, we could scan the field of possible hadronic molecules for dimer- and three-particle-candidates. An interesting application is the  $X(3872)$ -meson as a  $DD\pi$  system. Since the  $D\pi$ -dimer could only appear in  $p$ -wave scattering, higher derivatives in the fields would have to be included.

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## References

- [1] E. S. Swanson, Phys. Rept. **429** (2006) 243; E. Klempt and A. Zaitsev, Phys. Rept. **454** (2007) 1; M. B. Voloshin, Prog. Part. Nucl. Phys. **61** (2008) 455; S. Godfrey and S. L. Olsen, Ann. Rev. Nucl. Part. Sci. **58** (2008) 51.
- [2] M. B. Voloshin and L. B. Okun, JETP Lett. **23** (1976) 333 [Pisma Zh. Eksp. Teor. Fiz. **23** (1976) 369]; A. de Rujula, H. Georgi, and S. L. Glashow, Phys.

- Rev. Lett. **38** (1977) 317; N. A. Tornqvist, Phys. Rev. Lett. **67** (1991) 556; N. A. Tornqvist, Phys. Lett. B **590** (2004) 209; E. Braaten and M. Kusunoki, Phys. Rev. D **69** (2004) 074005.
- [3] D. V. Bugg, Phys. Lett. B **598** (2004) 8; C. Hanhart, Yu. S. Kalashnikova, A. E. Kudryavtsev and A. V. Nefediev, Phys. Rev. D **76**, 034007 (2007).
  - [4] C. Bignamini, B. Grinstein, F. Piccinini, A. D. Polosa and C. Sabelli, Phys. Rev. Lett. **103** (2009) 162001.
  - [5] P. Artoisenet and E. Braaten, P. Artoisenet and E. Braaten, Phys. Rev. D **81** (2010) 114018 [arXiv:0911.2016 [hep-ph]].
  - [6] J. D. Weinstein and N. Isgur, Phys. Rev. D **41** (1990) 2236; G. Janssen, B. C. Pearce, K. Holinde and J. Speth, Phys. Rev. D **52** (1995) 2690; C. Hanhart, Yu. S. Kalashnikova, A. E. Kudryavtsev and A. V. Nefediev, Phys. Rev. D **75** (2007) 074015; Yu. S. Kalashnikova, A. E. Kudryavtsev, A. V. Nefediev, C. Hanhart and J. Haidenbauer, Eur. Phys. J. A **24** (2005) 437.
  - [7] Yu. S. Kalashnikova, A. E. Kudryavtsev, A. V. Nefediev, J. Haidenbauer and C. Hanhart, Phys. Rev. D **78** (2008) 058501.
  - [8] N. N. Achasov and A. V. Kiselev, Phys. Rev. D **76** (2007) 077501.
  - [9] G. 't Hooft, G. Isidori, L. Maiani, A. D. Polosa and V. Riquer, Phys. Lett. B **662** (2008) 424 [arXiv:0801.2288 [hep-ph]].
  - [10] S. Weinberg, Phys. Rev. **137** (1965) B 672.
  - [11] V. Baru, J. Haidenbauer, C. Hanhart, Yu. Kalashnikova and A. E. Kudryavtsev, Phys. Lett. B **586** (2004) 53.
  - [12] S. R. Beane, P. F. Bedaque, W. C. Haxton, D. R. Phillips and M. J. Savage, in “At the Frontier of Particle Physics: Handbook of QCD: Boris Ioffe Festschrift”, M. Shifman (ed.), World Scientific (2001), arXiv:nucl-th/0008064; P. F. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. **52** (2002) 339.
  - [13] E. Braaten and H.-W. Hammer, Phys. Rept. **428** (2006) 259.
  - [14] F. K. Guo, C. Hanhart and U. G. Meißner, Phys. Lett. B **665** (2008) 26.
  - [15] P. Hagen, “Y(4660) in effektiver Feldtheorie”, Diploma thesis, University of Bonn (2010).
  - [16] P. F. Bedaque, H.-W. Hammer and U. van Kolck, Phys. Rev. Lett. **82** (1999) 463.
  - [17] F.-K. Guo, C. Hanhart and U.-G. Meißner, Eur. Phys. J. A **40** (2009) 171.
  - [18] L. Liu, H. W. Lin and K. Orginos, PoS **LATTICE2008** (2008) 112 [arXiv:0810.5412 [hep-lat]].
  - [19] C. Hanhart, Yu. S. Kalashnikova and A. V. Nefediev, Phys. Rev. D **81**, 094028 (2010).

- [20] D. V. Bugg, J. Phys. G **36** (2009) 075002.
- [21] G. Cotugno, R. Faccini, A. D. Polosa and C. Sabelli, Phys. Rev. Lett. **104** (2010) 132005 [arXiv:0911.2178 [hep-ph]].
- [22] F. K. Guo, J. Haidenbauer, C. Hanhart and U.-G. Meißner, arXiv:1005.2055 [hep-ph].